Noether Normalization: Given a variety $\emptyset \neq X \subseteq \mathbb{A}^n$, $\dim A(X) = m$,

$$\exists K[x_1,\dots,x_m] \overset{\text{module-finite}}{\hookrightarrow} A(X) \Rightarrow \text{morphism } P: X \to \mathbb{A}^m,$$

let $\underline{a} = (a_1,\dots,a_m) \in \mathbb{A}^m$, $I(\underline{a}) = (x_1 - a_1, \dots, x_m - a_m) \in \operatorname{Max}(K[x_1,\dots,x_m])$.

For $\underline{b} \in X$ let $\bar{I}(b) \in \operatorname{Max}(A(X))$ be the corresponding ideal.

$$\Rightarrow P^{-1}(\{\underline{a}\}) = \{\underline{b} \in X : \bar{I}(b) \cap K[x_1,\dots,x_m] = I(a)\} \quad (*)$$

(C7.6, T7.10, needs some thought!)

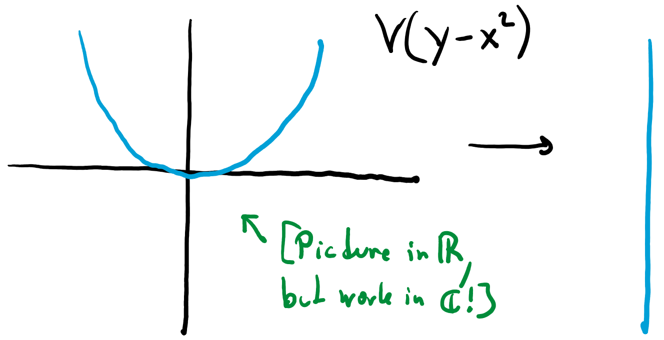
So: (1) If K alg. closed $\to P$ surjective (LYING OVER)

(2) P has finite fibers:

$K[x_1,\dots,x_m]/I(a) \hookrightarrow A(X)/I(a)A(X)$ is integral

$\Rightarrow \dim\left(A(X)/I(a)A(X)\right) = 0$. Since $A(X)/I(a)A(X)$ is also noetherian, it is ardinian (T4.14), hence has fin. many max. ideals (P4.12). By $(*)$ the fibers are finite.

Exm:



$F: \mathbb{A}^2 \to \mathbb{A}$

$(a,b) \mapsto b$

$$F^{-1}(\{b\}) = \begin{cases} \pm\sqrt{b} & \text{if } b \neq 0 \\ 0 & \text{if } b = 0 \end{cases}$$

generically $2:1$.

$$\mathbb{C}[y] \hookrightarrow \mathbb{C}[x,y]/(y-x^2) \quad \text{is integral}$$

b≠0: $\mathbb{C}[y]/(y-b) \hookrightarrow \mathbb{C}[x,y]/(y-b,\, y-x^2) = \mathbb{C}[x]/(x^2-b)$

$$= \mathbb{C}[x]/(x+b)(x-b) \underset{\text{CRT}}{\cong} \mathbb{C}[x]/(x+b) \times \mathbb{C}[x]/(x-b) \cong \mathbb{C} \times \mathbb{C}.$$

b=0: $\mathbb{C}[y]/(y) \hookrightarrow \mathbb{C}[x,y]/(y,\, y-x^2) = \mathbb{C}[x]/(x^2)$

Localization: $X \subseteq \mathbb{A}^n$ irred variety $\Rightarrow A(X)$ domain w. quotient field Q (rational functions on X)

Localization: X irred variety $\Rightarrow A(X)$ domain w. quotient field Q

$Q = \{\frac{f}{g} : f,g \in A(X), g \neq 0\}$. ← rational functions on X. Each $f \in A(X)$ is a function $f: X \to K$ (given by a polynomial in the coordinates, - continuous).

Let $\underline{a} := (a_1,\dots,a_n) \in X$, $M = (\bar{x}_1 - a_1, \dots, \bar{x}_n - a_n) \in \mathrm{Max}(A(X))$

Note: $f \in M \Leftrightarrow f(a_1,\dots,a_n) = 0$ [easy exc, cf. proof of T7.2]

$\Rightarrow A_M = \{\frac{f}{g} : f,g \in A(X), g(a_1,\dots,a_n) \neq 0\}$

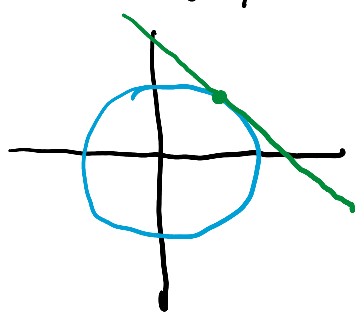
So A_M consists of those rational functions that are „well-defined" at $\underline{a}$, i.e, we can evaluate $\frac{f(\underline{a})}{g(\underline{a})}$. Since g is continuous (in Zariski-topology), $g^{-1}(\{0\})$ is closed, so $X \setminus g^{-1}(\{0\})$ is open and $\frac{f}{g}$ even induces a function on an open nbhd of $\underline{a}$.

$MA_M = \{\frac{f}{g} : g(\underline{a}) \neq 0, f(\underline{a}) = 0\}$ is the max. ideal of the local ring A_M.

7.3 Zariski (Co)tangent Space, Regularity

K field

Goal: Define tangent spaces of varieties algebraically (inspired by real analytic notions)



$R := K[x_1,\dots,x_n]$ $(n \geq 0)$.

Def: For $1 \leq i \leq n$, the formal derivative $\partial_{x_i} : R \to R$ is the unique K-linear map s.t.

K-linear map s.t.

$$\partial_{x_i}(x_j^n) = \begin{cases} n x_i^{n-1} & \text{if } n \geq 1,\ j=i \\ 0 & \text{if } j \neq i \text{ or } n=0 \end{cases}$$

Each $\partial = \partial_{x_i}$ is a derivation:

- K-linear: $\partial(af+bg) = a\,\partial(f) + b\,\partial(g)$ $\forall a,b \in K\ \forall f,g \in R$
- Leibniz rule: $\partial(fg) = \partial(f)g + f\,\partial(g)$ $\forall f,g \in R$

Let $\underline{a} = (a_1, \dots, a_n) \in \mathbb{A}^n$, $\underline{a} \in V(f)$

If $R \ni f = \sum_{i_1,\dots,i_n \geq 0} c_{i_1,\dots,i_n} (x_1 - a_1)^{i_1} \cdots (x_n - a_n)^{i_n}$ (*) (finite sum),

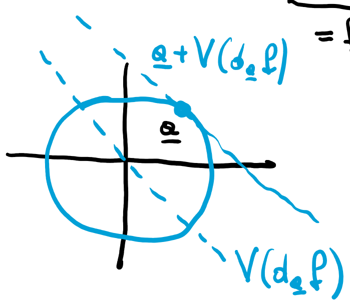
then $\partial_{x_i}(f)(\underline{a}) = c_{0,\dots,0,\underset{\substack{\uparrow \\ \text{place } i}}{1},0,\dots,0}$.

Think of $\sum_{i=1}^{n} \partial_{x_i}(f)(\underline{a})(x_i - a_i)$ as best $\overset{\text{affine}}{\text{linear}}$ approx. of $V(f)$ at $\underline{a}$. (affine space)

$d_{\underline{a}} f := \sum_{i=1}^{n} \partial_{x_i}(f)(\underline{a})\, x_i$ linear polynomial

$\Rightarrow$ $\underline{a} + V(d_{\underline{a}} f)$ is the affine linear approx of f at $\underline{a}$

Exm: $X = V(\underbrace{x^2+y^2-1}_{=f}) \subseteq \mathbb{A}^2(\mathbb{R})$ $\quad \partial_x f = 2x$, $\partial_y f = 2y$



$\underline{a} = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$, $\quad d_{\underline{a}} f = \frac{2}{\sqrt{2}} x + \frac{2}{\sqrt{2}} y$

Def: Let $X \subseteq \mathbb{A}^n$ be a variety, $\underline{a} \in X$. The tangent space $T_{\underline{a}} X$ of X at $\underline{a}$ is $V(\{d_{\underline{a}} f : f \in I(X)\})$.

Note: Since each $d_{\underline{a}} f$ is linear, this is a linear space.

Lemma 7.11: (1) If $I(X) = (f_1, \dots, f_r)$, then $T_{\underline{a}} X = V(d_{\underline{a}} f_1, \dots, d_{\underline{a}} f_r)$

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Lemma 7.11: (1) If $I(X) = (f_1, \dots, f_r)$, then $T_{\underline{a}} X = V(d_{\underline{a}} f_1, \dots, d_{\underline{a}} f_r)$

(2) If $J_{f_1,\dots,f_r}(\underline{a}) := \left(\partial_{x_j} f_i(\underline{a})\right)_{\substack{i=1,\dots,r \\ j=1,\dots,n}} \in K^{r \times n}$ is the Jacobian

then $T_{\underline{a}} X = \operatorname{Ker}\left(J_{f_1,\dots,f_r}(\underline{a})\right)$

Proof: (1) "$\subseteq$" ✓, "$\supseteq$":

Observe: $d_{\underline{a}}(fg) \overset{(*)}{=} d_{\underline{a}}(f) g(\underline{a}) + f(\underline{a}) d_{\underline{a}}(g) \overset{f(\underline{a})=0}{=} g(\underline{a})\, d_{\underline{a}} f \quad \forall g \in K[x_1, \dots, x_n]$

$$\Big[\ d_{\underline{a}}(fg) = \sum_{i=1}^{n} \partial_{x_i}(fg)(\underline{a})\, x_i = \sum_{i=1}^{n} \left(\partial_{x_i}(f) g + f \partial_{x_i}(g)\right)(\underline{a}) \cdot x_i$$

$$= \sum_{i=1}^{n} \Big[\partial_{x_i}(f)(\underline{a}) \underbrace{g(\underline{a})}_{\text{constant}} + \underbrace{f(\underline{a})}_{\text{constant}} \partial_{x_i}(g)(\underline{a})\Big] x_i = d_{\underline{a}}(f) g(\underline{a}) + f(\underline{a}) d_{\underline{a}}(g) \Big]$$

Let $f \in I(X) \to f = f_1 g_1 + \dots + f_r g_r$ with $g_i \in K[x_1, \dots, x_n]$

$\Rightarrow d_{\underline{a}} f = \sum_{i=1}^{r} g_i(\underline{a})\, d_{\underline{a}} f_i$ vanishes on $V(d_{\underline{a}} f_1, \dots, d_{\underline{a}} f_r)$.

(2) $\forall \underline{b} = (b_1, \dots, b_n) \in \mathbb{A}^n$: $\quad d_{\underline{a}} f_i(\underline{b}) = 0 \Leftrightarrow \sum_{j=1}^{n} (\partial_{x_j} f_i)(\underline{a}) \cdot b_j = 0$.

Prop 7.12 K field, $X \subseteq \mathbb{A}^n(K)$ variety, $\underline{a} = (a_1, \dots, a_n) \in X$, $R = K[x_1, \dots, x_n]$

Let $M = (\bar{x}_1 - a_1, \dots, \bar{x}_n - a_n) \in \operatorname{Max}(A(X))$. Then

$$T_{\underline{a}}^* X := (T_{\underline{a}} X)^* := \operatorname{Hom}_K(T_{\underline{a}} X, K) \cong M/M^2$$

cotangent space/space of differentials

Proof: Let $K[x_1, \dots, x_n]/I(X) = A(X)$, $f \mapsto \bar{f}$, $R = K[x_1, \dots, x_n]$

Define $\varphi: M \longrightarrow \operatorname{Hom}_K(T_a X, K)$, $\bar{f} \longmapsto d_a f|_{T_a X} = \left(\sum_{i=1}^{n} \partial_{x_i}(f)(\underline{a}) \cdot x_i\right)\Big|_{T_a X}$

Well-defined: If $g \in I(X)$, $d_{\underline{a}} g|_{T_a X} = 0$.

φ K-linear ✓

φ surjective: Let $\ell \in \operatorname{Hom}_K(T_a X, K)$. Extend ℓ linearly to $\mathbb{A}^n = K^n$:

$\Rightarrow \ell = \sum_{i=1}^{n} \lambda_i x_i$ for some $\lambda_i \in K$.

$\Rightarrow \ell = \sum_{i=1}^{n} \lambda_i x_i$ for some $\lambda_i \in K$.

Take $f = \sum_{i=1}^{n} \lambda_i (x_i - a_i) \Rightarrow \bar{f} \in M$, $d_a f = \ell$.

<u>$\operatorname{Ker}\psi = M^2$:</u> „$\supseteq$" $M^2 = {}_K\langle \bar{f}\bar{g} : f, g \in R$ s.t. $\bar{f}, \bar{g} \in M \rangle$

$\Rightarrow \psi(\bar{f}\bar{g}) = d_{\underline{a}}(fg)|_{T_{\underline{a}}X} \overset{*}{=} [d_{\underline{a}}(f)\underbrace{g(\underline{a})}_{=0} + \underbrace{f(\underline{a})}_{=0} d_{\underline{a}}(g)]|_{T_{a}X} = \underline{0}$.

$\Rightarrow M^2 \subseteq \operatorname{Ker}(\psi)$.

<u>„$\subseteq$"</u>: $L := \{ d_{\underline{a}} g : g \in I(X)\}$ is a subvectorspace of ${}_K\langle x_1, \dots, x_n \rangle$

Let $r := \dim_K L \leq n$. $\Rightarrow T_{\underline{a}}X = \{ \underline{b} \in \mathbb{A}^n : \forall \varphi \in L : \varphi(\underline{b}) = 0\}$ has dimension $n - r \to L' := \{ \varphi \in {}_K\langle x_1, \dots, x_n\rangle : \forall \underline{b} \in T_{\underline{a}}X : \varphi(\underline{b}) = 0 \}$ has dimension r. So $L \subseteq L' \to L' = L$.

Let $f \in R$ be s.t. $\bar{f} \in \operatorname{Ker}(\psi) \subseteq M$. So $d_{\underline{a}} f|_{T_{\underline{a}}X} = 0 \Rightarrow$ $d_{\underline{a}} f \in L' = L \Rightarrow \exists g \in I(X) : d_{\underline{a}} f = d_{\underline{a}} g \to d_a(f-g) = 0$.

$\Rightarrow \partial_{x_i}(f-g)(\underline{a}) = 0 \quad \forall i = 1, \dots, n, \qquad (f-g)(\underline{a}) = 0$

$\Rightarrow f - g = \sum_{\substack{i_1, \dots, i_n \geq 0 \\ i_1 + \dots + i_n \geq 2}} c_{i_1, \dots, i_n} (x_1 - a_1)^{i_1} \cdots (x_n - a_n)^{i_n} \in (x_1 - a_1, \dots, x_n - a_n)^2$.

$\Rightarrow \bar{f} - \bar{g} \in M^2$ and $\bar{g} = 0$, so $\bar{f} \in M^2$. $\square$

<u>Remark</u>: R ring, $M \in \operatorname{Max}(R)$, $S := R \setminus M \Rightarrow M/M^2$ is R/M-vector space

$\Rightarrow M/M^2 \cong S^{-1}(M/M^2) = S^{-1}M / S^{-1}M^2 = MR_M / M^2 R_M$.

<u>Def</u>: Let (R, M) be a local ring. The cotangent space of R is M/M^2. Its dual space over $K = R/M$ is the tangent space of R: $\operatorname{Hom}_K(M/M^2, K)$.

<u>Lemma 7.13</u> Let (R, M) be a noetherian local ring, $K = R/M$. Then $\dim_K M/M^2$ is the minimum number of generators for M.

Proof: Let $r := \dim M/M^2$. If $M = (a \quad a_-)$ then

then $\dim_K M/M^2$ is the minimum number of generators for M.

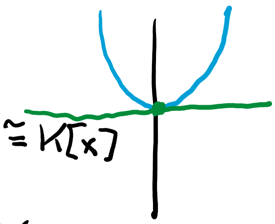
<u>Proof:</u> Let $r := \dim_K M/M^2$. If $M = (a_1, \dots, a_n)$, then

$M/M^2 = {}_K\langle \bar{a}_1, \dots, \bar{a}_n \rangle$, so $n \geq r$.

Let $b_1, \dots, b_r \in M$ s.t. $\bar{b}_1, \dots, \bar{b}_r$ is a K-basis of M/M^2.

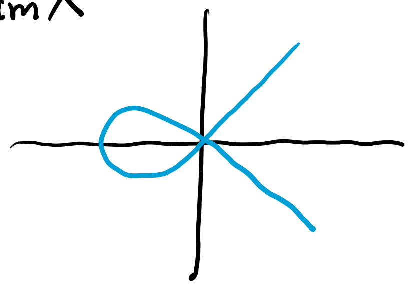
$\Rightarrow M = (b_1, \dots, b_r) + M \cdot M \xRightarrow{\text{Nakayama}} M = (b_1, \dots, b_r)$. $\square$

<u>Exm:</u> $K = \mathbb{R}$, $X = V(y - x^2) \subseteq \mathbb{A}^2$, $A = A(X) = K[x,y]/(y-x^2) \cong K[x]$



$\Rightarrow T_0 X = V(y)$

$(x,y)A = xA$, $\dim_{\mathbb{R}}\left(xA/x^2A\right) = 1 = \dim X$



$X' := V(y^2 - x(x + x^2))$

$y^2 - x^2 - x^3$

$T_0 X = V(0) = \mathbb{A}^2$

<u>$\dim_{\mathbb{R}}$</u> $(x,y)A_{(x,y)} / (x,y)^2 A_{(x,y)} = 2 > \dim(X') = 1$.